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Tensile Strength of Whiskers

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Tensile tests have been performed on whiskers of iron, copper, and silver 1.2 to 15 μ in diameter. The strongest whiskers which were less than 4 μ in diameter exhibited resolved elastic shear strengths of from two to six percent of their shear moduli. Stress-strain determinations on iron have shown that large deviations from Hooke's law occur beyond two percent strain. As the whiskers increase in size, their strengths decrease with considerable scatter.

INTRODUCTION

THE strength of crystals may be increased many fold when their size is made sufficiently small. This was first demonstrated clearly in 1924 by G. F. Taylor[1] in a paper on the preparation of metal wires in the micron size range. He reported the tensile strength of a 30 μ antimony wire to be 18 to 22 kg/mm^2 whereas the fracture strength of 4-mm antimony crystals[2] has been found to be between 0.57 and 0.77 kg/mm^2. It appears from the method of preparation of the wires that they were essentially single crystals.

Similarly the critical shear stress of cadmium crystals[3] etched down to 25 μ was found to be fourteen times higher than that of large crystals ($>500\ \mu$).

That crystals are potentially many times stronger than they normally are has been demonstrated even more emphatically by measurements on the recently discovered whiskers. For the first time single crystals in the micron size range became readily available. Bending tests on whiskers of Sn,[4] Fe,[5] ZnS,[6] Cu,[7] and Si[8] have shown that they can support elastic strains in excess of 10^{-2}. Bulk crystals rarely exhibit strains greater than 10^{-4}.

The tensile test is considerably better suited than the bend test for evaluating the strength of whiskers. Although the strain can be measured accurately in a bend test, the stress can only be estimated. Furthermore, the stress is complex and nonuniform both across and along the whiskers.

Tensile tests have been performed by Gyulai[9] on NaCl whiskers and by Eisner[10] on silicon whiskers. Gyulai in a very comprehensive study tested nearly 100 NaCl whiskers ranging from 1 to 25 μ in diameter. He found that the fracture strength is strongly dependent on size below 10 μ. A maximum strength of about 110 kg/mm^2 was reported.

Eisner fractured silicon whiskers in a pendulum type tensile apparatus and found a maximum strength of 390 kg/mm.[2] The size of the whiskers was reported to be in the micron and submicron range.

Both Gyulai and Eisner estimated the cross-sectional area of the whiskers by measuring their width. This method introduces some error because of the uncertainty of the orientation of the whiskers under the microscope and the possible variation in the cross-sectional shape from one whisker to another.

Because of the widespread interest in whiskers and their impact on modern theory of solids, the present study on the tensile strength of iron, copper, and silver whiskers was undertaken. The whiskers were grown by the reduction of their halides as described elsewhere.[7]

The paper is divided into two sections. In the first section the behavior of the strongest whiskers are reported and their strengths are compared with those expected of perfect crystals free of defects. The second section describes the effect of size and the presence of defects on the strength of whiskers.

EXPERIMENTAL

The tensile testing equipment is shown in Fig. 1. The whiskers 1.2 to 15 μ in diameter and 1 to 4 mm in

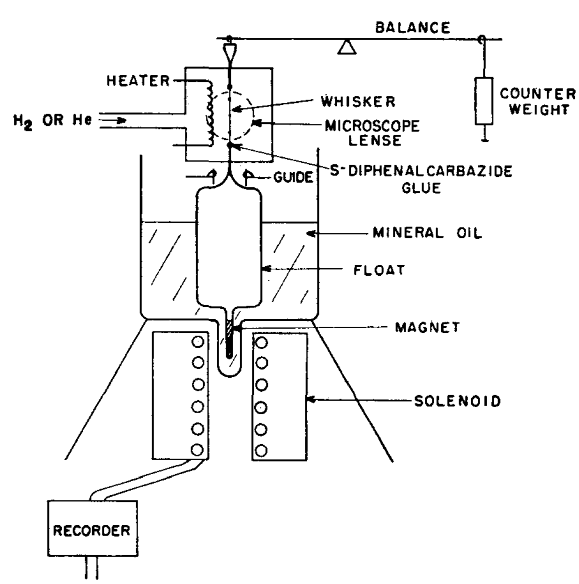

Fig. 1. Micro tensile machine.

[1] G. F. Taylor, Phys. Rev. **23**, 655 (1924).
[2] G. Wassermann, Z. Krist. **75**, 376 (1930).
[3] E. N. daC. Andrade, Institute of Metals Monograph No. 13, 138 (1953).
[4] C. Herring and J. K. Galt, Phys. Rev. **85**, 1060 (1952).
[5] Sears, Gatti, and Fullman, Acta Metallurgica **2**, 727 (1954).
[6] W. W. Piper and W. L. Roth, Phys. Rev. **92**, 503 (1953).
[7] S. S. Brenner, Acta Metallurgica **4**, 62 (1956).
[8] W. T. Read and G. L. Pearson (private communication).
[9] Z. Gyulai, Z. Physik **138**, 317 (1954).
[10] R. L. Eisner, Acta Metallurgica **3**, 419 (1955).

length were fastened from one end of the locked balance arm to a float resting in mineral oil. The float contained an Alnico magnet in the bottom and was positioned in a solenoid as shown. When the solenoid was activated the float was pulled down thus transmitting a stress to the whiskers. The force exerted by the solenoid was calibrated by using the balance in the normal unlocked position. Small changes in the position of the magnet did not alter appreciably the force on the float. The load required to fracture the whiskers ranged from 0.5 to 20 g and was applied at a rate varying from 0.2 to 1 g/min.

The whiskers were attached to the mounts by *S*-diphenyl carbazide (mp 173°C). To insure axial alignment the carbazide beads were softened momentarily

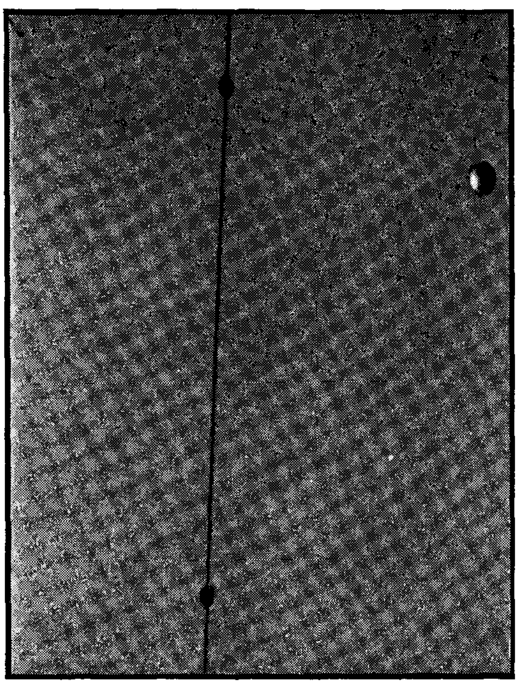

FIG. 2. Fiducial marks on strained whisker Mag$\sim$40$\times$.

while a small load was applied. With this procedure the final fracture of the whiskers usually occurred between the mounts. The whiskers were observed with a low power microscope during the test.

During mounting and testing the whiskers were surrounded with dry hydrogen or helium to minimize surface corrosion.

Two small carbazide beads on the whiskers as shown in Fig. 2 served as fiducial marks when the strain was measured. An enlarged image of the whisker was projected into a camera loaded with fine grained film and a picture of the whisker was taken after each increment of load. The developed film was then projected onto a screen and the change in length between the fiducial marks was measured with a cathetometer. The error in the strain measurements was estimated to be ± 0.001.

TABLE I. Tensile strength of whiskers.

1	2	3	4	Bulk crystals 5	6
Material	d (μ)	σ_{max} (kg/mm^2)	τ_{max} (kg/mm^2)	$\tau_{critical}$ (kg/mm^2)	Ultimate tensile strength (kg/mm^2)
Fe	1.60	1340	364	4.5[a]	16–23[b]
Cu	1.25	300	82	0.10[b]	12.9–35.0[b]
Ag	3.80	176	72	0.06[b]	

[a] See reference 12.
[b] See reference 13.

The cross-sectional area of the whiskers was determined by cross-sectioning them[11] and measuring the area at magnifications up to 2000$\times$. There was a considerable variation in the cross-sectional shape and the diameters of the whiskers are therefore defined for the purposes of this report, as the square root of the cross-sectional area. The calculated stress on the whiskers is probably accurate to within $\pm 10\%$ considering the errors in the area and force measurements.

The whiskers to be tested were selected by visual examination under a low power microscope. Only those were selected which had no apparent surface defects or overgrowth. Tapered whiskers were usually rejected.

The orientation of a few of the whiskers was determined. In general, the copper whiskers had their axes parallel to the [111] direction while the iron and silver whisker axes were parallel to the [100] direction. However, both the iron and copper whiskers exhibited axes parallel to all three principal directions [100], [110], and [111].

SECTION I—BEHAVIOR OF THE STRONGEST WHISKERS

Results—Tensile Strength

The tensile strength of more than 70 whiskers was determined with the major emphasis on copper and iron. The smallest whiskers exhibited the highest strengths. The maximum strength of the whiskers and their size are given in Table I. "Strength" is here defined as the maximum stress the whisker sustained before fracture occurred. As shown later this strength is equivalent to the stress at the elastic limit.

Column 3 of Table I shows the calculated resolved shear stresses, τ_{max}, corresponding to the measured

TABLE II. Shear strength of whiskers.

1	2	3	4	5	6
Whiskers	Whisker axes	Slip system	G(kg/mm^2)	τ_{max}/G	τ_{th}/G (estimated)
Fe	[111]	(110) [111]	6100	0.060	0.033–0.19
Cu	[111]	(111) [101]	3700	0.022	0.033–0.13
Ag	[100]	(111) [101]	2300	0.031	0.033–0.13

[11] S. S. Brenner and C. R. Morelock (to be published).
[12] W. Fahrenhorst and E. Schmid, Z. Physik 78, 383 (1932). (See footnote to Table I.)

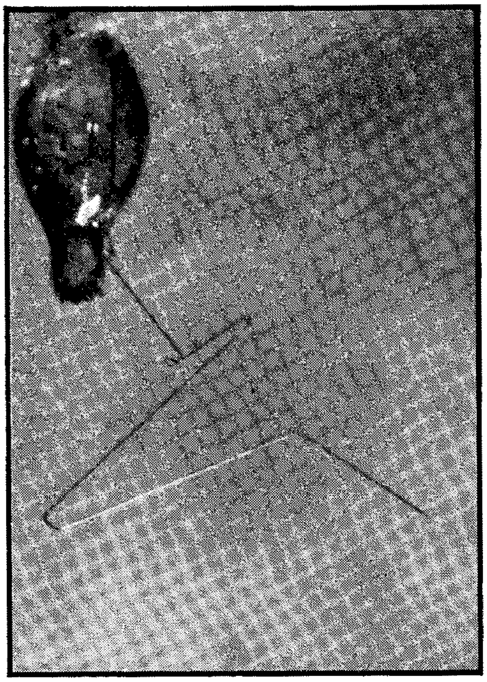

FIG. 3. Appearance of iron whisker after fracture at a stress above 250 kg/mm². Mag 25×.

tensile stresses. The slip systems employed to obtain the resolved shear stresses are indicated in Table II.

A comparison with the reported strengths of bulk crystals (columns 5 and 6) shows that the yield point shear strengths of the whiskers are 80 to 1200 times greater than those of large crystals. The ratios of the ultimate tensile strengths are lower.

Mode of Fracture

The Cu and Fe whiskers described in Table I fractured in a spectacular manner. They suddenly snapped without any observable amount of plastic deformation and wrapped themselves around as shown in Fig. 3. The silver whisker, however, exhibited considerable plastic deformation prior to fracture.

Stress-Strain Relationship

The true stress-true strain curves of two iron whiskers are shown in Fig. 4. Whisker *I* is the same as the one described in Table I. Both of the whiskers were first stressed and unstressed before the final data was taken. Whisker *I* was initially stressed to almost the final yield stress, while whisker *II* was stressed to about half the final yield stress. No plastic deformation was apparent after the initial loading and unloading, and the strain shown in Fig. 4 is believed to be completely elastic.

Within the experimental accuracy Hooke's law is obeyed up to about two percent strain. Beyond that, the strain increases more rapidly than stress—Young's modulus is no longer a constant. A maximum strain of 0.048 was measured. The moduli as calculated from the initial slopes are 30 000 kg/mm² and 18 500 kg/mm², respectively. The Young's modulus of iron is reported[13] to range from 13 500 kg/mm² for the [100] direction to 29 000 kg/mm² for the [111] direction. Since no x-ray diffraction was taken from the two iron whiskers one must conclude that the axis of whisker *I* was probably parallel to the [111] direction while the axis of whisker *II* was either parallel to the [100] or [110] direction.

To check the accuracy of the method, stress-strain measurements were made on a quartz fiber. A similar determination was previously made by Reinkober[14] who found the stress to be proportional to the strain up to fracture. As Fig. 4 shows, no deviation from Hooke's law was found with the quartz fiber. The fracture strength of the 4.3 μ fiber was 420 kg/mm² and the modulus 8750 kg/mm². These values agree well with those found by Reinkober for 4.3 μ fibers.

Effect of Temperature

All tensile testing was performed at room temperature. Several preliminary elevated temperature tests were made with copper whiskers. The whiskers were maintained in a looped position and heated in vacuum at a pressure of about 10^{-6} mm Hg. The whiskers were strained 0.002 to 0.003. In one test a whisker, 5.4 μ diameter, was strained to 0.0025 ($\sigma \simeq 50$ kg/mm² with [111] axis) and maintained at 900°C for two hours. The whisker did not show any permanent set after the restraint was removed.

Discussion: Comparison of the Theoretical Shear Strength of Perfect Crystals and the Maximum Measured Strengths

The theoretical tensile strength σ_{th} of crystals has been satisfactorily calculated only in the case of ionic crystals such as NaCl. Zwicky,[15] considering the electrostatic forces between the ions calculated σ_{th} to be 200 kg/mm². De Boer[16] taking into account also the Van der Waals attraction arrived at 400 kg/mm². Gyulai's measured maximum strength of 110 kg/mm² for NaCl whiskers is in fair agreement with the calculated σ_{th}.

Because of the nature of the binding forces in metals, the theoretical strength of metal crystals has as yet not been satisfactorily calculated. An estimate of the theoretical shear stress, τ_{th}, required to nucleate slip in the absence of dislocations has been made by several authors* using the following model. Two rows of atoms

[13] E. Schmid and W. Boas, *Plasticity of Crystals* (F. A. Hughes and Company, Ltd., London).
[14] O. Reinkober, Physik. Z. **32**, 243 (1932); Physik. Z. **33**, 32 (1932).
[15] F. Zwicky, Physik. Z. **24**, 131 (1923).
[16] J. H. De Boer, Trans. Faraday Soc. **32**, 10 (1936).
* See reference 18 for detailed review.

in the shear plane and shear direction such as shown in Fig. 5 are considered. The distance between the rows is "a" while the interatomic distance in the slip direction is "b". The force required to shear the two rows of atoms or to move an atom from the A position to the equivalent C position is calculated. At $x=0$ and $x=b$ the force required to disturb the atom is zero. At position B or $x=b/2$ the atom is in a metastable position and the shearing force again becomes zero. The force therefore varies in a periodic fashion. Frenkel[17] assumed a sinusoidal function such as $\tau=k\sin(2\pi x/b)$. When x is small Hooke's law, $\tau=Gx/a$, where G is the elastic shear modulus in the shear direction, is obeyed and k can be evaluated. The critical shear stress or τ_{th} occurs at $b/4$ and is

$$\tau_{\text{th}}=bG/a2\pi.$$

The critical shear strain $b/4$ seemed unreasonably high. In a modification of the above model Mackenzie[18] proposed a force function such as curve 2 in Fig. 5. He concluded, however, that τ_{th} could not be much less than $G/30$.

Bragg and Lomer[19] by means of bubble raft studies also came to the conclusion that the theoretical shear strength of perfect metal crystals, in particular, copper is about $G/30$. They simulated a perfect copper crystal by adjusting the bubble size in their raft so that the elastic interactions between the soap bubbles was comparable to that between copper atoms. The critical strain to initiate shear in the bubble raft corresponded to a shear strength of $G/30$.

The present estimates of the theoretical strengths of metal crystals then lie between $G/30$ and $(b/a)(G/2\pi)$. The highest resolved shear strengths of iron, copper,

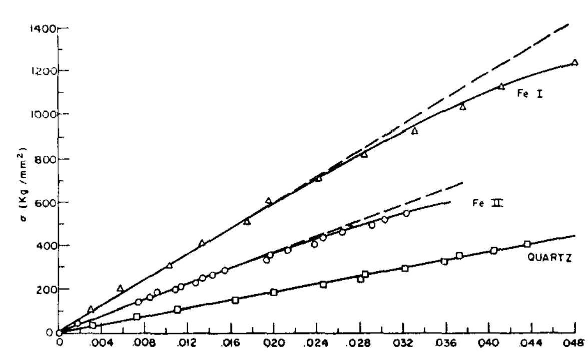

FIG. 4. Stress-strain curves of iron and quartz:

Fe I—1.6 μ diam	Y_0 = 30 000 kg/mm².
Fe II—3.8 μ diam	Y_0 = 18 500 kg/mm².
Quartz—4.3 μ diam	Y_0 = 8 750 kg/mm².

(Y_0 = Young's modulus evaluated from initial slope).

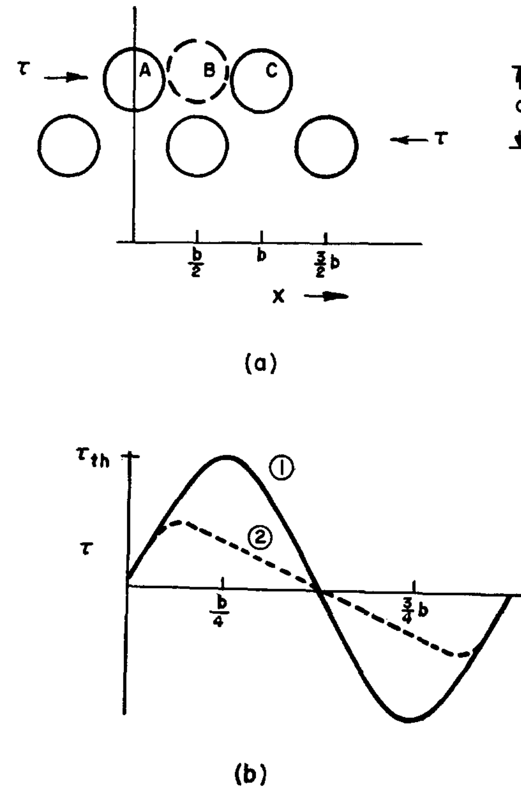

FIG. 5. (a) Shear of two rows of atoms in the slip plane along the slip direction. (b) Shearing force acting during slip.

and silver in terms of G as determined in this investigation are compared with those estimated on theoretical grounds in Table II. The slip systems used to resolve the tensile stresses are given in column 3. G was calculated from

$$1/G=S_{44}+4(S_{11}-S_{12}-1/2S_{44})(\alpha\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2),$$

where S_{ih} = shear moduli[13] and α, β, γ = direction cosines of the shear directions.

As Table II indicates, the highest shear strengths of the whiskers are either close to or above the lower estimate of the theoretical strength of perfect crystals.

Stress-Strain Relationship

The shape of the stress-strain curves of the two iron whiskers in Fig. 4 also appears to indicate that the theoretical shear strength was closely approached. If the shearing force varies approximately as depicted by curve 2 in Fig. 5, *large* deviations from Hooke's law will only be detected near τ_{th}. The difference between $\sin 2\pi x/b$ and $2\pi x/b$ is less than 7% at $x=0.1b$.

Deviations from Hooke's law became noticeable as shown by Fig. 4 at $\epsilon\simeq 0.02$. The deviations may, however, have started at lower strains and escaped detection because of the decreased accuracy of the strain measurements at lower strains.

[17] J. Frenkel, Z. Physik **37**, 572 (1926).
[18] A. H. Cottrell, *Dislocations and Plastic Flow in Crystals* (Clarendon Press, Oxford, 1953), p. 9.
[19] W. L. Bragg and W. M. Lomer, Proc. Roy. Soc. (London) **A196**, 171 (1949); W. M. Lomer, *ibid.* **A196**, 182 (1949).

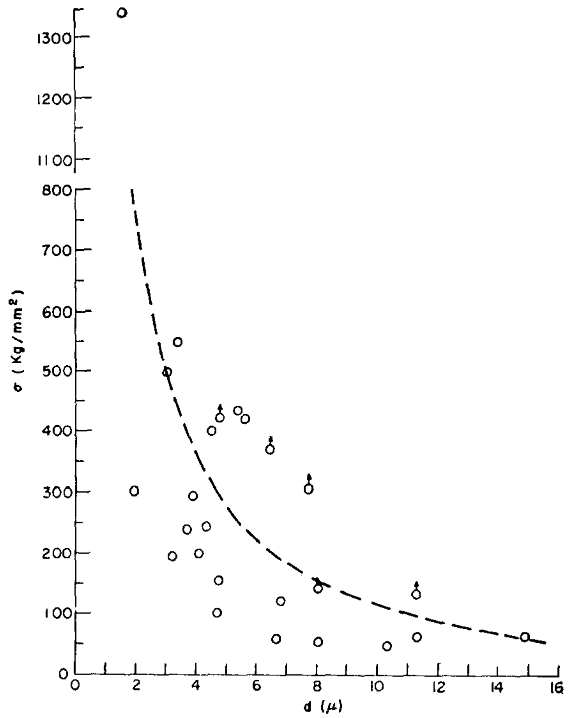

FIG. 6. The effect of size on the strength of iron whiskers. ♁ fracture occurred at or near grips. True fracture stress may have been higher.

SECTION II—EFFECT OF SIZE AND DEFECTS ON THE STRENGTH OF WHISKERS

Results—Effect of Size-Diameter

The effect of size on the strength of iron, copper, and silver whiskers is shown in Figs. 6–8. Although there is considerable scatter, a marked size effect is apparent especially in the case of iron and copper.

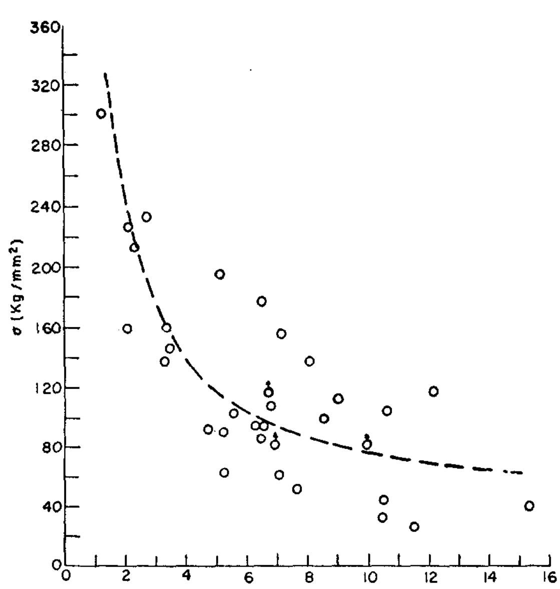

FIG. 7. The effect of size on the strength of copper whiskers.

In spite of the scatter there is an indication that the *average* strength is inversely proportional to the diameter. In Figs. 9 and 10 the average strength of groups of five whiskers of nearly the same size is plotted against the reciprocal of their average diameter. The average strength σ_{Av} as evaluated from Figs. 9 and 10 can be described adequately by

$$\sigma_{Av}=\frac{410}{d}+36 \text{ kg/mm}^2 \text{ copper,} \quad (1)$$

$$\sigma_{Av}=\frac{1630}{d}-50 \text{ kg/mm}^2 \text{ iron,} \quad (2)$$

where d is in microns.

Equation (2) can obviously be valid only for the size range studied (1–15 μ) since σ_{Av} becomes negative as d increases. The average strengths corresponding to (1) and (2) are indicated in Figs. 6 and 7 by the dashed lines.

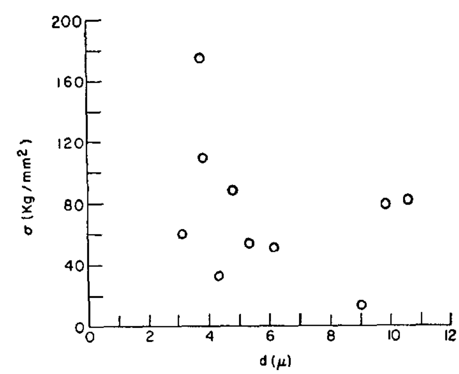

FIG. 8. The effect of size on the strength of silver whiskers.

It is of interest to observe that even at 15 μ the iron and copper whiskers still possessed unusually high elastic strengths (Fe—65 kg/mm², Cu—38 kg/mm²).

Effect of Size—Length

A number of times when a whisker broke one half was reconnected and restressed. In almost all instances the strength of the whisker increased as the length decreased. Table III gives one example of an iron whisker of 4.71 μ diameter and about 4 mm in length which was fractured and reconnected three times. As the length of the whisker was decreased, the strength increased by a factor greater than four.

Effect of Surface Conditions

The surface condition of the whiskers greatly influenced their strengths. The surface defects encountered were (1) pits, (2) overgrowth and whisker branches, (3) spotty corrosion product, and (4) occasional grain boundaries. The strength of the whiskers containing these defects was invariably low. The most effective deteriorating agent was plasticene placed in the vicinity of the whiskers.

Continuous oxide films formed on the whiskers by heating in air at 100 to 150°C did not change their strength seriously. The strength of vacuum-heated whiskers appeared to be somewhat higher. This effect will be studied further.

Mode of Fracture

The mode of fracture depended on the material and stress level. No plastic deformation of iron whiskers ever was observed with a low powered microscope. They fractured without warning, and if the stress was sufficiently high, they would snap back and wrap themselves around as described previously. Examination of the fractured ends at high magnification sometimes revealed a small amount of necking while at other times the fracture seemed to start from surface cracks. Typical examples of fractures are shown in Fig. 11.

Some of the copper whiskers failed in a manner similar to that of the iron whiskers. Many, however,

TABLE III.

Sequence of test	Fracture strength (kg/mm²)	Location of point of fracture
1st fracture	99	Between grips
2nd fracture	153	Between grips
3rd fracture	423	At grip

exhibited appreciable amounts of plastic deformation. Just prior to fracture the whiskers would suddenly yield at one or more points with the appearance of slip lines which were easily visible on the highly reflectant whisker surfaces. The deformation front would then run along the whisker and fracture would finally occur. The duration of the plastic flow was much less than one second. The flow stress which was measured occasionally was always lower than the stress at yielding.

All of the silver whiskers tested failed with considerable prior deformation as described above for copper.

Because of the sharp yield points exhibited by the whiskers it seems reasonable to assume that the strengths measured in this investigation were in the elastic region.

Discussion: Effect of Size on the Strength of the Whiskers

The very large scatter in the strength of whiskers as a function of their size indicates that the strengths of the perfect whiskers must be decreased by defects which are distributed statistically in a rather complex manner. These defects initiate premature fracture either with or without large amounts of prior deformation. The possible variation in the orientation and length of the whiskers also may contribute to the scatter to some extent.

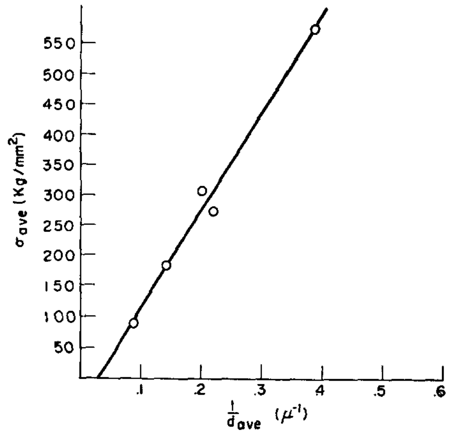

FIG. 9. The average strength of iron whiskers as a function of the reciprocal of the diameter.

The defects are believed to be introduced accidentally during the growth of the whiskers and are not considered to be inherent to the crystals. The defect density is therefore a function of the mode of growth. By changing such factors as the conditions of growth, purity of materials, etc. it should be possible to alter the total number of defects and possibly their distribution and thus grow large crystals essentially free of defects. Equations (1) and (2) therefore apply only to the strength of whiskers grown under a particular set of circumstances.

The defects responsible for the decrease in strength are presumably located both on the surface and in the interior of the whiskers. Coarser surface defects can be detected by observing the reflectivity of the whisker surfaces. When the reflection of light from the whisker surfaces is nonuniform, the strength is invariably low. It is believed that surface defects below the resolution

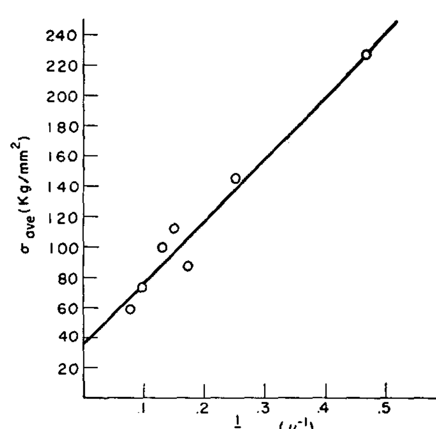

FIG. 10. The average strength of copper whiskers as a function of the reciprocal of the diameter.

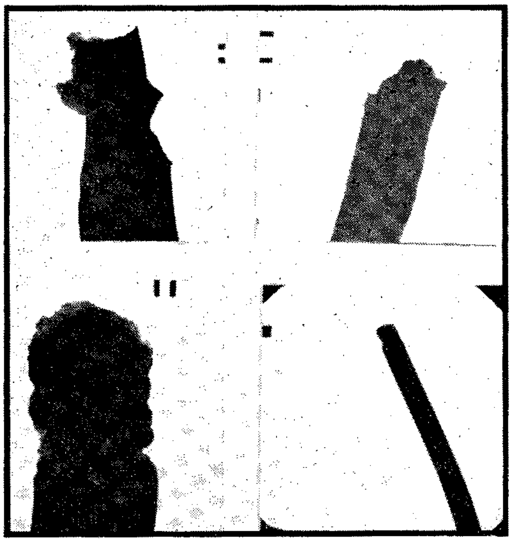

FIG. 11. Appearance of fractures: Upper left Fe whisker fractured at 160 kg/mm², 1870×. Upper right Cu whisker fractured at 168 kg/mm², 2500×. Lower left Fe whisker fractured at 130 kg/mm², 1870×. Lower right Fe whisker fractured at 182 kg/mm², 500×.

of light microscopy contribute considerably toward decreasing the strength.

The internal defects are probably dislocation sources such as postulated by Frank and Read. The resolved shear stress required to activate this type of dislocation source is given as

$$\tau = Gb/l,$$

where l = length of the pinned dislocation segment and b = Burger's vector of the dislocation segment. In the case of copper and iron the length of the dislocation sources in the most probable slip planes could not have been much greater than 0.1 μ for any of the whiskers tested. This is only a small fraction of their diameters.

A statistical analysis of the strength is difficult to make. Factors to be considered are the nature, orientation, and location of the defects, effectiveness and distribution of the defects, and the orientation and size of the crystals. In view of the limited data a statistical analysis was not attempted.

The linear dependence of the average strength on $1/d$ does not indicate whether the defects are located on the surface of the crystals or in the interior. The number of surface defects of a given effectiveness is expected to vary as d and the strength may therefore be proportional to $1/d$. Likewise, if the whiskers contain a number of internal dislocation sources of varying size the length of the longest source may also vary as d and the strength as $1/d$. If the decrease in strength is due to the chance formation of a *single* internal dislocation source during the growth, the strength is expected to be inversely proportional to the cross-sectional area of the whisker or $1/d^2$.

The effect of the length on the strength of the whiskers emphasizes again that fracture is initiated at singular defects. These defects must be small in number and sparsely distributed. Under these circumstances the scatter in the data is expected to be considerable.[20]

The large amount of plastic flow exhibited by many of the copper whiskers and all of the silver whiskers may indicate that after the crystal slips, because of the operation of a single defect, new dislocation sources are created or activated by the continued plastic flow.

Other Strengthening Effects

There are two additional factors that may contribute to the strength of whiskers (1) impurities and (2) surface films. Impurities can strengthen the whiskers by pinning the few dislocation sources that may be present. At least in copper, pinning does not seem to contribute a major proportion of the strengthening. If it did, a strong temperature and time dependence of the strength would be expected which, at least qualitatively, was not found for copper in the high temperature bending tests described in Sec. I. Elevated temperature tensile tests are at present being conducted to verify this point more thoroughly. Impurities may also help to weaken the whiskers by promoting the formation of dislocations during the growth of the whiskers.

It has been reported that surface films increase the strength of crystal. Roscoe[21] found that oxide films (detected by interference colors) increased the critical shear stress of cadmium crystals by as much as 0.072 kg/mm² or about 150% the normal critical shear stress. The thicker the oxide, the greater was the strengthening effect. This "Roscoe effect" was later confirmed by Cottrell and Gibbons.[22]

Roscoe further observed that the strengthening effect of the oxide film increases as the diameter of the crystal is decreased. Contrary to Roscoe's result, Makin[3] found that "clean" and oxide-coated cadmium crystals increased in strength with decreasing diameter in the same fashion.

The scatter in the strength of whiskers of the same size and the effect of length on the strength also seem to indicate that surface films do not contribute appreciably to the strength of the whiskers. Since it is unlikely that NaCl has a coherent surface film, Gyulai's work on NaCl whiskers is further evidence against the importance of surface films.

Summary

It has been shown that the strongest iron, copper, and silver whiskers exhibit strengths which are either

[20] J. C. Fisher and J. H. Hollomon, T. P. 2218, Am. Inst. Mining Met. Engrs. (1947).
[21] R. Roscoe, Phil. Mag. **21**, 399 (1936).
[22] A. H. Cottrell and D. F. Gibbons, Nature **162**, 488 (1948).

close to or above the lower estimate of the strength of perfect crystals. The stress-strain curves of two of the strongest iron whiskers indicate that within the experimental accuracy Hooke's law is obeyed up to about two percent strain. Beyond two percent strain, considerable deviation from linearity occurs. The stress-strain behavior is taken as a sign that the theoretical strength of perfect crystals has been closely approached.

The strongest whiskers were also the smallest in size. As the diameter and length is increased, the strength of the whiskers decreases with considerable scatter. It is postulated that this decrease in strength is due to defects which are formed accidentally during the growth of the whiskers.

ACKNOWLEDGMENTS

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Electromagnetic Transmission Characteristics of a Lattice of Infinitely Long Conducting Cylinders*

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This investigation is primarily concerned with the study of electromagnetic transmission characteristics of a lattice of infinitely long conducting cylinders. Four approaches to the general problem have been employed and the advantages, disadvantages, and realms of validity of each have been studied. Several of these methods constitute a substantial improvement over previous analyses and are supported by wave-guide and free-space experimental work.

The four approaches are: (1) A molecular analogy with a consideration of dipole interactions leading to the Clausius-Mosotti relations, (2) A transmission line formulation which considers the thick obstacle for both polarizations, (3) An analysis based on the summation of scattered fields which demonstrates that the Clausius-Mosotti relations are a special case of a more general relation which accounts for the effects of higher order multipoles, and (4) A solution formally valid for all values of spacing and cylinder radius based on integral equation formulation of variational principle.

1. INTRODUCTION

THIS investigation is concerned with the electromagnetic transmission properties of periodic structures of infinitely long conducting cylinders. The practical application of artificial media as microwave lenses and polarizing devices has produced great interest in the transmission properties of lattices consisting of different geometrical shapes, such as disks, strips, etc. The analytic treatment of such problems has been carried along two general lines:

(a) Molecular and model analysis, and

(b) Cellular methods.

The molecular approach is based on the assumption that each element acquires electric and magnetic dipole moments under the action of an incident field. The analysis is then analogous to that employed in the study of actual dielectrics with the elements taking the place of diamagnetic molecules. The essential restrictions under which such an analogy is meaningful are that (a) the size of the conducting element is small compared to the lattice spacings since, in general, interaction effects can be predicted to only a limited extent, particularly for nonisotropic obstacles, (b) the largest dimension of the element is small compared to wavelength because the analysis assumes that induced dipole moments are computed on a static basis, and finally (c) the lattice spacings are small compared to wavelength since the theory cannot predict Bragg reflections.

The modal analysis introduced by Brown[1] and Cohn[2] considers the lattice as a stack of parallel plate wave guides, each wave guide periodically loaded along its length by the lattice obstacles. Although a formal analysis is possible for unrestricted obstacle size and lattice spacing, practicality imposes the restriction that the transverse lattice spacing is such as to permit only single mode propagation and longitudinal lattice spacing great enough to insure only dominant mode interaction.

In the cellular method used by Wigner-Seitz and Slater, the solution of the electromagnetic field problem for the infinite lattice is equivalently reduced to the solution for the fields in a unit cell surrounding an element. In addition to the boundary conditions imposed by perfectly conducting obstacles, the fields must

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